Last time, we showed the equivalence

solving a system $\iff$ solving a matrix equation

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases} \iff Ax = b$$

$\Updownarrow$

determining if b lies in $\operatorname{span}\{A_1, \dots, A_n\}$

where $A = \begin{pmatrix} A_1 & \dots & A_n \end{pmatrix}$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$$= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

(the meaning of "equivalence" above is that all three problems lead to the same set of solutions x)

We saw a Theorem which said that $\exists$ solution

$\forall\, b \in \mathbb{R}^m$ if and only if REF of A has no all-zero rows

$$Ax = b$$

rows > # columns

$$A = \begin{pmatrix} 1 & * & 0 & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad b \in \mathbb{R}^6$$

Any such matrix A must have at least one all-0 row

$\Downarrow$

$\exists$ b's for which $Ax=b$ inconsistent

rows < # columns

$$A = \begin{pmatrix} \boxed{1} & * & * & 0 & * & 0 \\ 0 & 0 & 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix} \qquad b \in \mathbb{R}^3$$

free variables

equation $Ax=b$ never has 1 solution, b/c $\exists$ free variables

<u>Today</u>: we will study solution sets of $Ax=b$

$$\{x \in \mathbb{R}^n \text{ such that } Ax = b\}$$

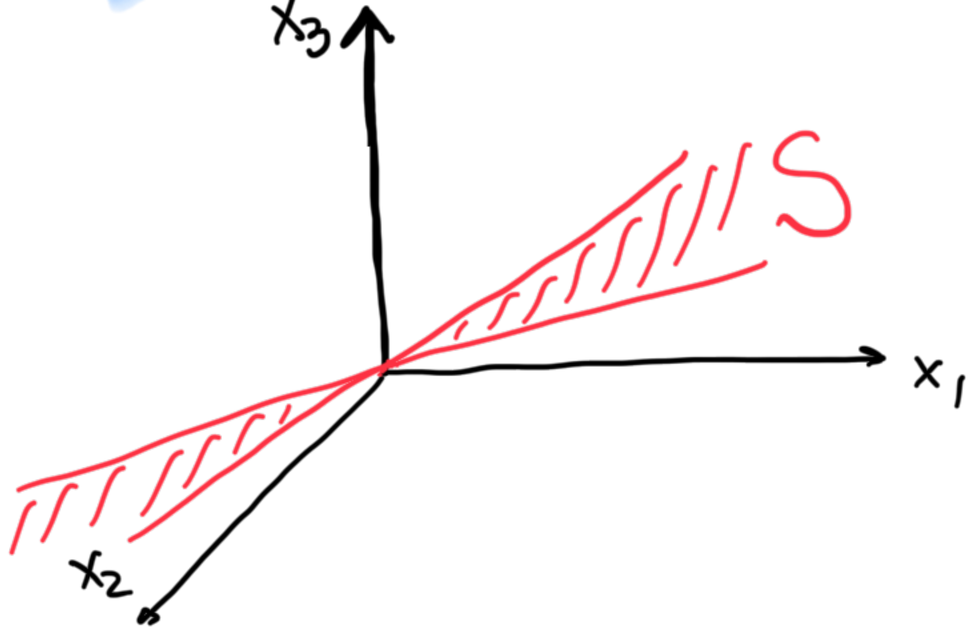
geometrically

two types of matrix equations
- homogeneous $Ax = 0$
- inhomogeneous $Ax = b$ (non-zero)

Homogeneous equations: $Ax = 0$

$$\begin{pmatrix} 2 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2x_1 + 3x_2 - 7x_3 = 0$$

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \text{ such that } 2x_1 + 3x_2 - 7x_3 = 0 \right\}$$



$\forall$ homogeneous equation $Ax = 0$, S is a $\underbrace{\text{subspace of } \mathbb{R}^n}_{\text{lines, planes etc which pass through } 0}$

- why does $S \ni 0$? Because $A0 = 0$.
- if $v, w \in S$, then $cv + dw \in S$, $\forall c, d \in \mathbb{R}$

Proof:

$$\left.\begin{aligned} Av = 0 &\Rightarrow cAv = 0 \\ Aw = 0 &\Rightarrow dAw = 0 \end{aligned}\right\} \Rightarrow cAv + dAw = 0$$

$$\Downarrow$$

$$A(cv + dw) = 0$$

General (including inhomogeneous) equations

$$Ax = b$$

(the associated homogeneous equation is $Ax = 0$)

THM 1: any solution $A \in S$ of

THM 4.1: Any solution s of the equation $Ax=b$ is of the form

$$s = s_{particular} + s_{homogeneous}$$

(s_b) — one fixed solution of $Ax=b$

(s_h) — an arbitrary solution of $Ax=0$

In other words, $\exists$ one-to one correspondence

$$\left\{\begin{array}{c}\text{solutions } s \text{ of}\\ Ax=b\end{array}\right\} \underset{s=s_b+s_h \;\leftarrow\; s_h}{\overset{s \;\to\; s_h=s-s_b}{\rightleftarrows}} \left\{\begin{array}{c}\text{solutions } s_h \text{ of}\\ Ax=0\end{array}\right\}$$

Example: $\begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$

Gaussian elimination

$$\begin{pmatrix} \boxed{1} & 0 & -\frac{4}{3} \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

free

$x_3 = t$

$x_2 = 2$

$x_1 - \frac{4}{3}x_3 = -1 \implies x_1 = \frac{4t}{3} - 1$

$$S=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\frac{4t}{3}-1\\2\\t\end{pmatrix},\ t\in\mathbb{R}\right\}$$

separate out free variables from each other and from numbers

$$=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-1\\2\\0\end{pmatrix}+t\begin{pmatrix}\frac{4}{3}\\0\\1\end{pmatrix},\ t\in\mathbb{R}\right\}$$

parametric vector form

$S_{particular}=S_p$

$S_{homogeneous}=S_h$

Do the same for the associated homogeneous equation

$$\begin{pmatrix}3&5&-4\\-3&-2&4\\6&1&-8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Gaussian elimination

$$\begin{pmatrix}\boxed{1}&0&-\frac{4}{3}\\0&\boxed{1}&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

free

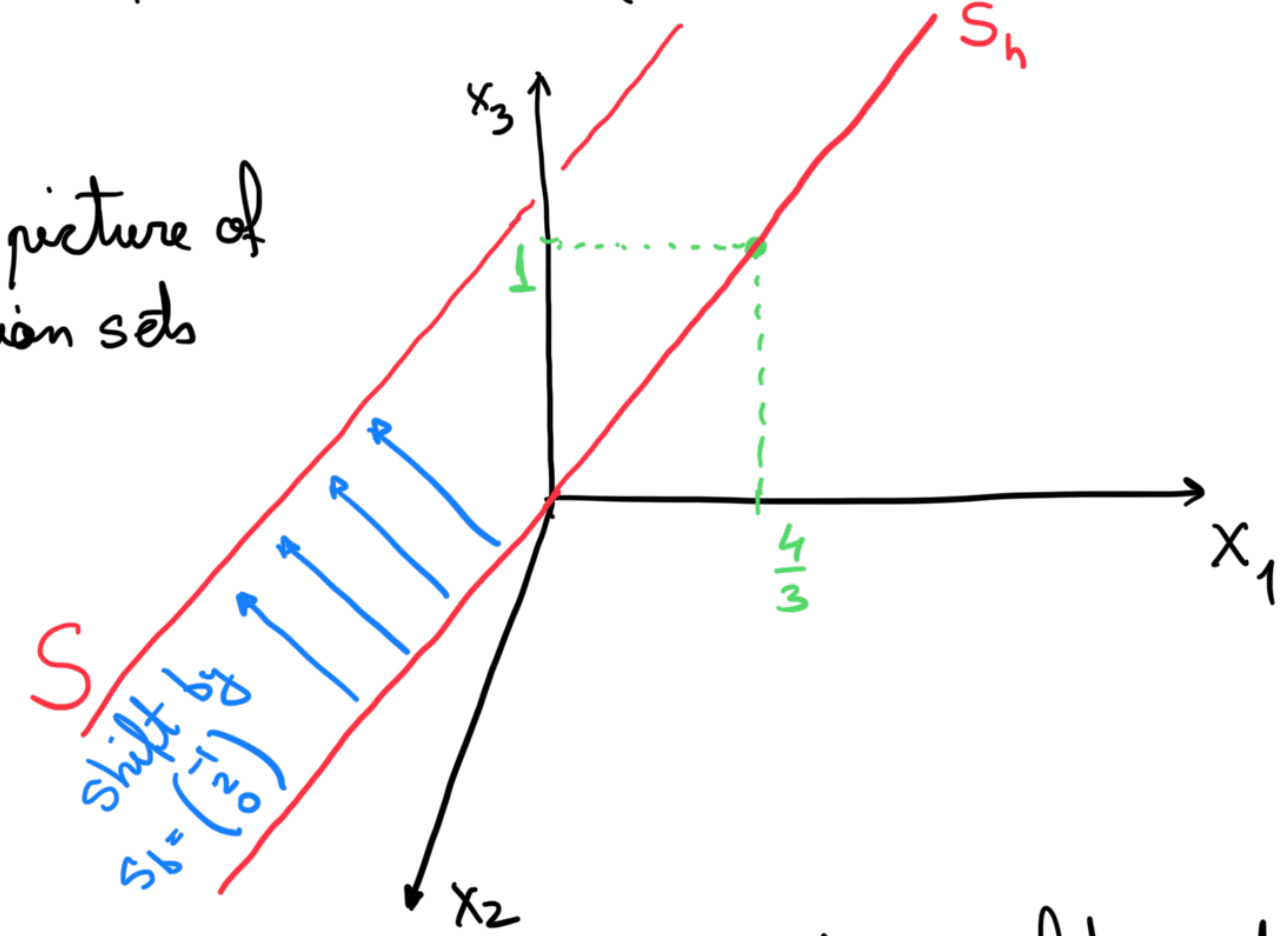
$$x_3=t$$

$$x_2=0$$

$$x_1-\frac{4}{3}x_3=0\implies x_1=\frac{4}{3}t$$

$$S_h = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}, \ \forall t \in \mathbb{R} \right\}$$

Geometric picture of above solution sets



Upshot: solution set to $Ax=b$ is obtained from the solution set to $Ax=0$ by shifting with the vector s_b (i.e. the particular solution)

$\Downarrow$

Solution set is always a (shifted) subspace

Principle for solving $Ax=b$ in parametric vector form

- $(A \mid b) \xrightarrow{\text{Gaussian elimination}} (A^{REF} \mid b^{REF})$
- set the free variables equal to parameters $t, s, u, \ldots$
- use the REF to solve for the basic/pivot variables

e.g. $$S=\left\{\begin{pmatrix} 7+9t-3s+4u+\dots \\ 6-\quad 2s+19u+\dots \\ 16t+\pi s+\sqrt{3}\,u+\dots \end{pmatrix}\right\}\ \forall\ t,s,u,\dots\ \mathbb{R}$$

- separate out the parameters from each other and from numbers

$$S=\left\{\underbrace{\begin{pmatrix}7\\6\\0\end{pmatrix}}_{S_b\text{ (particular solution)}}+\underbrace{t\begin{pmatrix}9\\0\\16\end{pmatrix}+s\begin{pmatrix}-3\\2\\\pi\end{pmatrix}+u\begin{pmatrix}4\\19\\\sqrt{3}\end{pmatrix}+\dots}_{S_h\text{ (homogeneous)}}\quad t,s,u\in\mathbb{R}\right\}$$

Example $$\begin{pmatrix}2 & 4 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}8\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}2 & 4 & -1 & 8\end{array}\right)\xrightarrow[\text{elimination}]{\text{Gaussian}}\left(\begin{array}{ccc|c}\boxed{1} & 2 & -\frac{1}{2} & 4\end{array}\right)$$

free

$x_3=t$

$x_2=s$

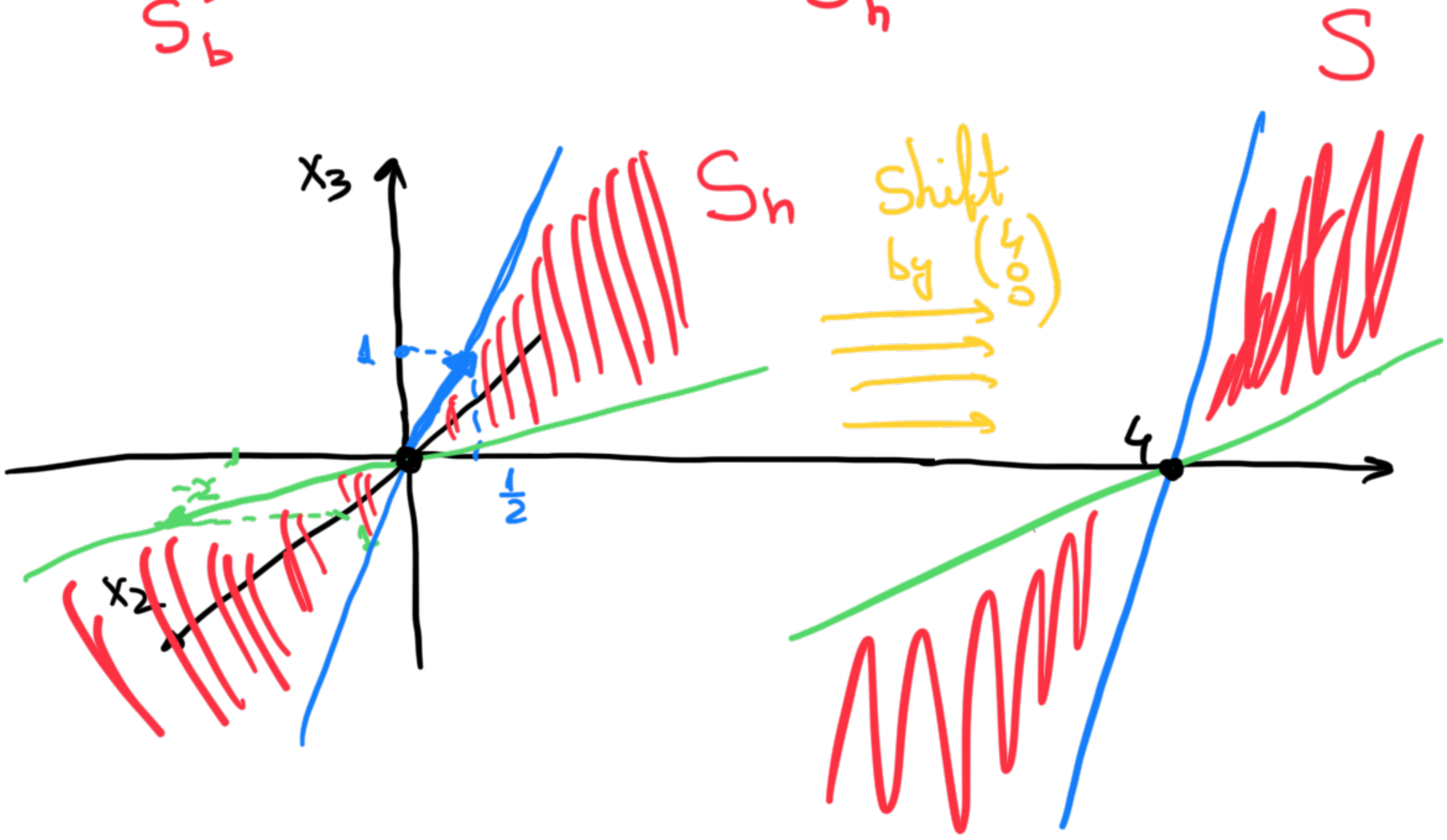
$$x_1+2x_2-\frac{x_3}{2}=4\implies x_1=4-2s+\frac{t}{2}$$

$$S=\left\{\begin{pmatrix}4-2s+\frac{t}{2}\\ s\\ \vdots\end{pmatrix},\quad\forall s,t\in\mathbb{R}\right\}$$

t

$$S = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \underbrace{s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}}_{S_h} \quad \forall s,t \in \mathbb{R} \right\}$$

S_b



Recall Thm: The solution set for $Ax = b$

$$S = \{ s = s_b + s_h \}$$

particular

arbitrary solution of $Ax=0$

<u>Proof</u>: $\subseteq$ → included in

$\supseteq$ → includes

free to choose

$\subseteq$: ... any solution s can be written as $s_b + s_h$

$\subseteq$: must prove any solution s can ...

given fixed

$$\begin{aligned} As &= b \\ As_b &= b \end{aligned} \implies As - As_b = b - b = 0 \implies A(s - s_b) = 0$$

so $s_h := s - s_b$ is a solution to homogeneous equation, and $s = s_b + s_h$

$\supseteq$: must show any sum of the form $s = s_b + s_h$ is also a solution to $Ax = b$

particular homogeneous

$$\begin{aligned} As_b &= b \\ As_h &= 0 \quad (+) \\ \hline As_b + As_h &= b \end{aligned}$$

$$\Downarrow$$

$$A(s_b + s_h) = b \implies s := s_b + s_h \text{ is also solution to } Ax = b$$

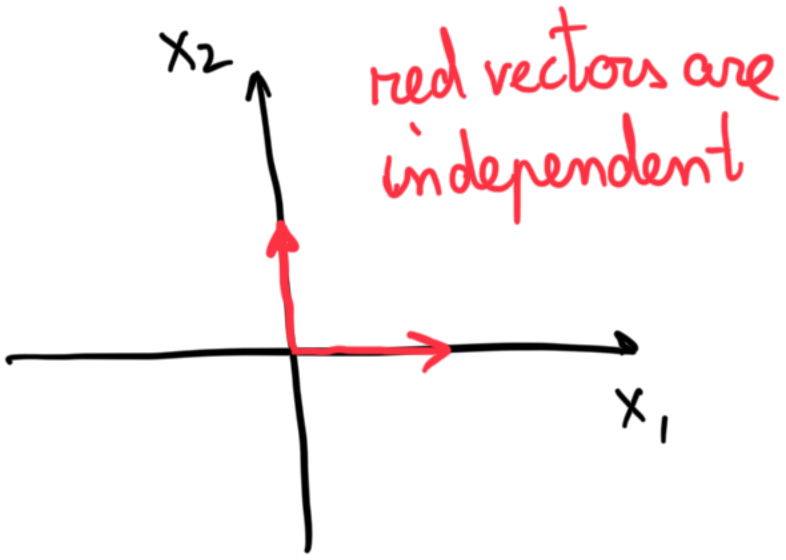
Linear (in)dependence of vectors

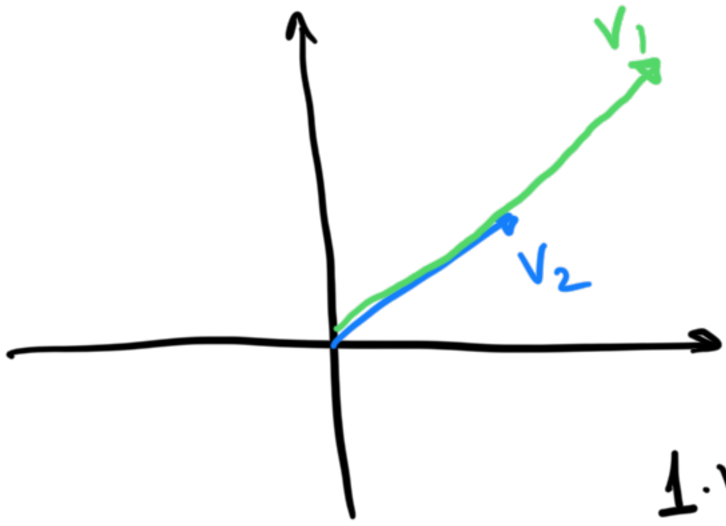
DEF 4.2: a set $v_1, \ldots, v_m \in \mathbb{R}^n$ are called **linearly dependent** if $\exists\, c_1, \ldots, c_m \in \mathbb{R}$ s.t.

not all 0

$$\boxed{c_1 v_1 + \dots + c_m v_m = 0}$$

They are **linearly independent** otherwise.

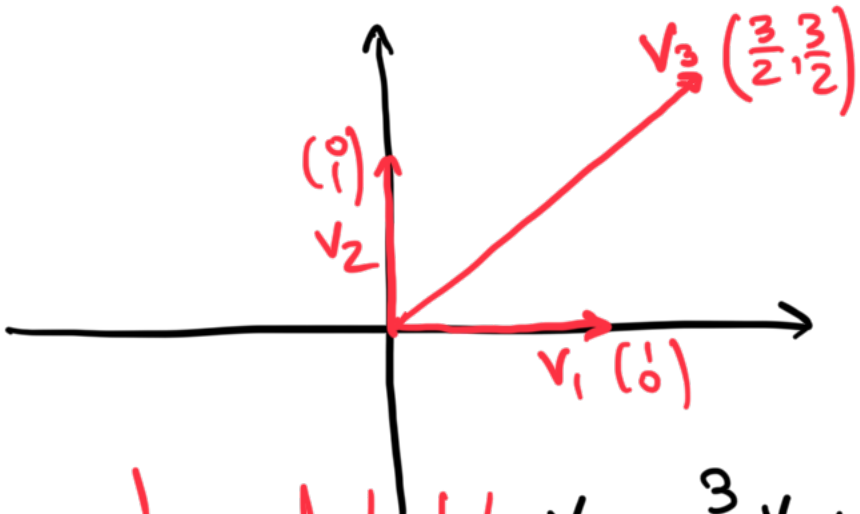




green and blue vectors are dependent b/c $v_1 = 2v_2$

$$\Downarrow$$

$$1 \cdot v_1 + (-2)v_2 = 0$$



red vectors are dependent b/c $v_3 = \frac{3}{2}v_1 + \frac{3}{2}v_2 \Rightarrow \frac{3}{2}v_1 + \frac{3}{2}v_2 + (-v_3) = 0$

Geometric meaning: $v_1, \dots, v_m$ are linearly independent $\Longleftrightarrow$ span$\{v_1, \dots, v_m\}$ has dim m

How to find out if vectors $v_1, \dots, v_m$ are linearly (in)dependent?

$$\downarrow$$

$Ax = 0$, where $\Longleftrightarrow$ $\exists\, c_1, \dots, c_m$ s.t. $c_1 v_1 + \dots + c_m v_m = 0$

$A = \left(v_1 \mid v_2 \mid \dots \mid v_m \right)$

$X = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$

Upshot:

independent if $Ax=0$ has just 0 as a solution

dependent if $Ax=0$ has ∞ many solutions

Next time

Thm 1.7 in exercise sheet

THM 5.1: let $v_1, \dots, v_m$ be vectors in $\mathbb{R}^n$

they are linearly dependent

$\Updownarrow$

one of the v_j's is a linear combination of the others

Thm 1.8 in exercise sheet

Prop:
- if $m > n$, then any set of vectors $\{v_1, \dots, v_m\}$ are linearly dependent
- any set of vectors among whom is 0 (i.e. $\{v_1, \dots, v_{k-1}, 0, v_{k+1}, \dots, v_m\}$) are linearly dependent

Thm 1.9 in exercise sheet

Prop: Given vectors $v_1, \dots, v_m, w \in \mathbb{R}$

- if $\{v_1, \dots, v_m\}$ are linearly dependent
  then $\{v_1, \dots, v_m, w\}$ are linearly dependent

if $\{v_1, \dots, v_m, w\}$ are linearly independent,
then $\{v_1, \dots, v_m\}$ are linearly independent